

# A YAU PROBLEM FOR VARIATIONAL CAPACITY

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**ABSTRACT.** Through using the semidiameter (in connection to: the mean radius; the  $p - 1$  integral mean curvature radius; the graphic ADM mass radius) of a closed convex hypersurface in  $\mathbb{R}^n$  with  $n \geq 2$  as an sharp upper bound of the variational  $p$  capacity radius, this paper settles an extension of S.-T. Yau's [71, Problem 59] from the surface area to the variational  $(1, n) \ni p$  capacity whose limiting as  $p \rightarrow 1$  actually induces the surface area.

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## 1. THEOREM AND ITS COROLLARY

In his problem section of Seminar on Differential Geometry published by Princeton University Press 1982, S.-T. Yau raised the following problem (cf. [71, Page 683, Problem 59]):

*Let  $h$  be a real-valued function on  $\mathbb{R}^3$ . Find (reasonable) conditions on  $h$  to insure that one can find a closed surface with prescribed genus in  $\mathbb{R}^3$  whose mean curvature (or curvature) is given by  $h$ .*

Since posed, this problem has received a lot of attention – see also: [63, 1, 72, 32, 19] for the aspect of mean curvature; [52, 53, 6, 16, 65, 66, 64, 9, 69] for the aspect of Gauss curvature; [30, 29] and their references for the aspect

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of curvature measure. In this paper, we study the above problem with genus zero from the perspective of the so-called variational  $p$  capacity. To be more precise, it is perhaps appropriate to review F. Almgren's comments on the Yau's problem (see the mid part of [71, Page 683, Problem 59]):

*For "suitable"  $h$  one can obtain a compact smooth submanifold  $\partial A$  in  $\mathbb{R}^3$  having mean curvature  $h$  by maximizing over bounded open sets  $A \subset \mathbb{R}^3$  the quantity*

$$F(A) = \int_A h d\mathcal{L}^3 - \text{Area}(\partial A).$$

*A function  $h$  would be suitable, for example, in case it were continuous, bounded, and  $\mathcal{L}^3$  summable, and  $\sup F > 0$ . However, the relation between  $h$  and the genus of the resulting extreme  $\partial A$  is not clear.*

Note that  $\text{Area}(\partial A)$  is just the variational 1-capacity of  $\partial A$  whenever  $A$  is convex body, i.e.,  $A \in \mathbb{K}^3$ , where  $\mathbb{K}^n$  comprises all compact and convex subsets of  $\mathbb{R}^n$  with nonempty interior (cf. [44], [23] and [47, Page 149]). So, as a variant of the Yau problem, it seems interesting to consider the maximizing problem below:

$$\sup \left\{ F_{\text{pcap}}(A) = \int_A h d\mathcal{L}^n - \text{pcap}(A) : A \in C^n \right\}.$$

In the above and below,  $C^n$  stands for the class of all compact and convex subsets of  $\mathbb{R}^n$  (the  $2 \leq n$ -dimensional Euclidean space) and  $\text{pcap}(E)$  is the variational  $1 \leq p < n$  capacity of an arbitrary set  $E \subset \mathbb{R}^n$ :

$$\text{pcap}(E) = \inf_{\text{open } U \supseteq E} \text{pcap}(U) = \inf_{\text{open } U \supseteq E} \left( \sup_{\text{compact } K \subseteq U} \text{pcap}(K) \right),$$

where for a compact subset  $K \subset \mathbb{R}^n$  one uses

$$\text{pcap}(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla f|^p d\mathcal{L}^n : f \in C_0^\infty(\mathbb{R}^n) \text{ \& } f \geq 1_K \right\},$$

with  $d\mathcal{L}^n$  denoting the usual  $n$ -dimensional Lebesgue measure and  $1_K$  being the characteristic function of  $K$ .

According to [31, Page 32], we have  $\text{pcap}(A) = \text{pcap}(\partial A)$  provided that  $A \subset \mathbb{R}^n$  is compact. This yields

$$1\text{cap}(A) = \text{Area}(\partial A) \quad \forall \quad A \in \mathbb{K}^n.$$

Physically speaking,  $2\text{cap}(A)$  of a compact set  $A \subset \mathbb{R}^3$  expresses the total electric charge flowing into  $\mathbb{R}^3 \setminus A$  across the boundary  $\partial A$  of  $A$ . Moreover, in accordance with Colesanti-Salani's calculation in [14] we see that for  $p \in (1, n)$  the capacity  $\text{pcap}(A)$  of  $A \in \mathbb{K}^n$  can be determined via

$$(1.1) \quad \text{pcap}(A) = \int_{\mathbb{R}^n \setminus A} |\nabla u_A|^p d\mathcal{L}^n = \int_{\partial A} |\nabla u_A|^{p-1} d\mathcal{H}^{n-1},$$

where  $d\mathcal{H}^{n-1}$  represents the  $(n-1)$ -dimensional Hausdorff measure on  $\partial A$ ,  $u_A$  is the so-called  $p$ -equilibrium potential, i.e., the unique weak solution to the following boundary value problem:

$$(1.2) \quad \begin{cases} \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 & \text{in } \mathbb{R}^n \setminus A; \\ u = 1 & \text{on } \partial A \quad \& \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \end{cases}$$

and the vector  $\nabla u_A$  exists almost everywhere as the non-tangential limit on  $\partial A$  with respect to  $d\mathcal{H}^{n-1}$ ; see also Lewis-Nyström's [42, Theorems 3-4].

Below is the main result of this paper.

**Theorem 1.1.** *Given  $p \in (1, n)$ ,  $\alpha \in (0, 1)$  and a nonnegative integer  $k$ , let  $h$  be a positive, continuous and  $L^1$ -integrable function on  $\mathbb{R}^n$ .*

(i)  *$F_{pcap}(\cdot)$  attains its supremum over  $C^n$  if and only if there exists  $A \in C^n$  such that  $F_{pcap}(A) \geq 0$ .*

(ii) *Suppose  $A \in \mathbb{K}^n$  is a maximizer of  $F_{pcap}(\cdot)$ . Then such an  $A$  satisfies the variational Eikonal  $p$ -equation  $(p-1)|\nabla u_A|^p = h$  in the sense of*

$$(1.3) \quad \int_{\mathbb{S}^{n-1}} \phi \mathbf{g}_*((p-1)|\nabla u_A|^p d\mathcal{H}^{n-1}) = \int_{\mathbb{S}^{n-1}} \phi \mathbf{g}_*(h d\mathcal{H}^{n-1}) \quad \forall \phi \in C(\mathbb{S}^{n-1}),$$

where  $\mathbf{g}_*(X d\mathcal{H}^{n-1})$  is the push-forward measure of a given nonnegative measure  $X d\mathcal{H}^{n-1}$  via the Gauss map  $\mathbf{g}$  from  $\partial A$  to the unit sphere  $\mathbb{S}^{n-1}$  of  $\mathbb{R}^n$ :

$$\mathbf{g}_*(X d\mathcal{H}^{n-1})(E) = \int_{\mathbf{g}^{-1}(E)} X d\mathcal{H}^{n-1} \quad \forall \text{ Borel set } E \subset \mathbb{S}^{n-1},$$

with  $\mathbf{g}^{-1}$  being the inverse of the Gauss map  $\mathbf{g}$ . In particular, if  $\partial A$  is  $C^2$  strictly convex, then  $(p-1)|\nabla u_A|^p = h$  holds pointwisely on  $\partial A$ .

(iii) *If  $h$  is of  $C^{k,\alpha}$  and  $A$ , with  $\partial A$  being  $C^2$  strictly convex, is a maximizer of  $F_{pcap}(\cdot)$ , then  $\partial A$  is of  $C^{k+1,\alpha}$ .*

Theorem 1.1 can actually give much more information than just a generalized solution to the Yau problem for  $pcap(\cdot)$  with  $1 < p < n$ . To see this, recall two related facts. The first is:

$$(1.4) \quad \operatorname{div}(|\nabla u|^{p-2} \nabla u) = |u_\nu|^{p-2}((n-1)Hu_\nu + (p-1)u_{\nu\nu}),$$

where  $\nu$ ,  $u_\nu$ ,  $u_{\nu\nu}$ , and  $H$  denote the outer unit normal vector, the first-order derivative along  $\nu$ , the second-order derivative along  $\nu$ , and the mean curvature of the level surface of  $u$  respectively, and so,

$$\operatorname{div}(|\nabla u|^{-1} \nabla u) = ((n-1)H)\left(\frac{u_\nu}{|u_\nu|}\right)$$

holds at least weakly. The second is Maz'ya's isocapacitary inequality for  $p \in [1, n)$  (cf. [46]):

$$(1.5) \quad \left( \frac{\mathcal{L}^n(E)}{\omega_n} \right)^{\frac{1}{n}} \leq \left( \left( \frac{p-1}{n-p} \right)^{p-1} \left( \frac{\text{pcap}(E)}{\sigma_{n-1}} \right) \right)^{\frac{1}{n-p}} \quad \forall \quad E \subset \mathbb{R}^n$$

and Federer's isoperimetric inequality (cf. [21, §3.2.43]):

$$(1.6) \quad \left( \frac{\mathcal{L}^n(E)}{\omega_n} \right)^{\frac{1}{n}} \leq \left( \frac{\mathcal{H}^{n-1}(\partial E)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}} \quad \forall \quad E \in C^n.$$

Here and henceforth,  $\omega_n$  and  $\sigma_{n-1} = n\omega_n$  stand for the volume and the surface area of the unit ball of  $\mathbb{R}^n$  respectively. Of course, the equality in (1.5)/(1.6) holds as  $A$  is a ball. Moreover, the left hand side of (1.5)/(1.6) is called the volume radius of  $E$ , and the right hand sides of (1.5) and (1.6) are called the variational  $p$  capacity radius and the surface radius respectively.

Now, our issue is as follows - the treatment of Theorem 1.1 brings not only Corollary 1.2 - a generalized solution to a special case (i.e., genus = 0) of the original Yau problem, but also a new analytic approach to some related geometric problems.

**Corollary 1.2.** *Let  $h \in L^1(\mathbb{R}^n)$  be positive and continuous,  $k$  be a nonnegative integer,  $\alpha \in (0, 1)$ , and*

$$F_{\mathcal{H}^{n-1}}(A) = \int_A h d\mathcal{L}^n - \mathcal{H}^{n-1}(A) \quad \forall \quad A \in C^n.$$

- (i)  $F_{\mathcal{H}^{n-1}}(\cdot)$  attains its supremum over  $C^n$  if and only if there exists  $A \in C^n$  such that  $F_{\mathcal{H}^{n-1}}(A) \geq 0$ .
- (ii) Suppose  $A \in \mathbb{K}^n$  is a maximizer of  $F_{\mathcal{H}^{n-1}}(\cdot)$ . Then there is a Borel measure  $\mu_{\mathcal{H}^{n-1}, A}$  on  $\mathbb{S}^{n-1}$  such that  $d\mu_{\mathcal{H}^{n-1}, A} = \mathbf{g}_*(h d\mathcal{H}^{n-1})$ , namely,

$$(1.7) \quad \int_{\mathbb{S}^{n-1}} \phi d\mu_{\mathcal{H}^{n-1}, A} = \int_{\mathbb{S}^{n-1}} \phi \mathbf{g}_*(h d\mathcal{H}^{n-1}) \quad \forall \quad \phi \in C(\mathbb{S}^{n-1}).$$

*In particular, if  $\partial A$  is  $C^2$  strictly convex, then such a maximizer  $A$  satisfies the equation  $h(\cdot) = H(\partial A, \cdot)$  - the mean curvature of  $\partial A$ .*

- (iii) *If  $h$  is of  $C^{k, \alpha}$  and  $A$ , with  $\partial A$  being  $C^2$  strictly convex, is a maximizer of  $F_{\mathcal{H}^{n-1}}(\cdot)$ , then  $\partial A$  is of  $C^{k+2, \alpha}$ .*

## 2. FIVE LEMMAS AND THEIR PROOFS

In order to prove Theorem 1.1 and Corollary 1.2, we will not only keep in mind the iso-capacitary/isoperimetric inequality (1.5)/(1.6) which shows that the volume radius serves as a sharp lower bound of the variational  $p$  capacity radius and the surface radius, but also explore the optimal upper bounds of these two geometric quantities in terms of the semidiameter, the

mean radius, the  $p-1$  integral mean curvature radius, and the graphic ADM mass radius; see the coming-up next five lemmas. In short, under certain conditions on  $A$  (and its boundary  $\partial A$  as well as its interior  $A^\circ$ ),  $1 < p < n$ ,  $H(\partial A, \cdot)$  and  $f$ , we will build the following decisive radius tree

$$\left( \frac{\mathcal{L}^n(A)}{\omega_n} \right)^{\frac{1}{n}} \leq \begin{cases} \left( \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right)^{\frac{1}{n-p}} \right) \\ \left( \frac{\left( \frac{p-1}{n-p} \right)^{1-p}}{\left( \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}}} \right) \end{cases} \leq \begin{cases} \frac{\text{diam}(A)}{2} \\ \frac{\mathbf{b}(A)}{2} \\ \left( \frac{1}{\sigma_{n-1}} \int_{\partial A} (H(\partial A, \cdot))^{p-1} d\mathcal{H}^{n-1}(\cdot) \right)^{\frac{1}{n-p}} \\ (2m_{ADM}(\mathbb{R}^n \setminus A^\circ, \delta + df \otimes df))^{\frac{1}{n-2}}, \end{cases}$$

and surprisingly find that if all principal curvatures of a given  $C^2$  boundary  $\partial A$  are in the interval  $[\alpha, \beta] \subset (0, \infty)$  then

$$\left( \frac{p-1}{(n-p)\beta} \right)^{p-1} \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right) \leq \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \leq \left( \frac{p-1}{(n-p)\alpha} \right)^{p-1} \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right).$$

**2.1. Semidiameter.** The isodiameter or Bieberbach's inequality (cf. [20, Page 69] and [57, Page 318]) says that the diameter  $\text{diam}(A)$  of  $A \subset \mathbb{R}^n$  dominates the double of the volume radius of  $A$ :

$$(2.1) \quad \left( \frac{\mathcal{L}^n(A)}{\omega_n} \right)^{\frac{1}{n}} \leq \frac{\text{diam}(A)}{2}$$

with equality if  $A$  is a ball. Interestingly, (2.1) has been improved through the foregoing (1.5)/(1.6) and the following (2.2)/(2.3).

**Lemma 2.1.**

(i) If  $p \in (1, n)$  and,  $A \subset \mathbb{R}^n$  is a connected compact set, then

$$(2.2) \quad \left( \left( \frac{p-1}{n-p} \right)^{p-1} \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right) \right)^{\frac{1}{n-p}} \leq \frac{\text{diam}(A)}{2}$$

holds, where equality is valid as  $A$  is a ball.

(ii) If  $A \in C^n$ , then

$$(2.3) \quad \left( \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}} \leq \frac{\text{diam}(A)}{2}$$

holds, where equality is valid as  $A$  is a ball.

*Proof.* Obviously, equalities in (2.2) and (2.3) occur when  $A$  is a ball. Note that (2.3) is the well-known Kubota inequality (cf. [39, 45]). So, it suffices

to prove the remaining part of (2.2). To do so, suppose

$$\begin{cases} \text{dist}(x, A) = \inf_{y \in A} |x - y|; \\ r\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < r\} \quad \forall \quad r > 0; \\ \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}; \\ S(A, t) = \mathcal{H}^{n-1}(\{x \in r\mathbb{B}^n \setminus A : \text{dist}(x, A) = t\}) \quad \forall \quad t > 0. \end{cases}$$

The flat case of Gehring's Theorem 2 in [25] implies that if

$$A \subset r\mathbb{B}^n \quad \& \quad \tau = \liminf_{x \rightarrow \overline{\mathbb{R}^n} \setminus r\mathbb{B}^n} \text{dist}(x, A),$$

then

$$(2.4) \quad \text{pcap}(r\mathbb{B}^n, A) \leq \left( \int_0^\tau (S(A, t))^{\frac{1}{1-p}} dt \right)^{1-p},$$

where

$$\text{pcap}(r\mathbb{B}^n, A) = \inf_u \int_{r\mathbb{B}^n \setminus A} |\nabla u|^p d\mathcal{L}^n$$

for which the infimum ranges over all functions  $u$  that are continuous in  $\overline{\mathbb{R}^n}$  and absolutely continuous in the sense of Tonelli in  $\mathbb{R}^n$  with  $u = 0$  in  $A$  and  $u = 1$  in  $\overline{\mathbb{R}^n} \setminus r\mathbb{B}^n$ .

Noting such an essential fact that if  $\hat{A}$  is the convex hull of  $A$  then

$$\text{pcap}(A) \leq \text{pcap}(\hat{A}) \quad \& \quad \text{diam}(A) = \text{diam}(\hat{A}),$$

without loss of generality we may assume that  $A$  is convex, and then restate Kubato's inequality (cf. [39, 27]) for such an  $A$ :

$$\frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \leq \left( \frac{\text{diam}(A)}{2} \right)^{n-1}.$$

This in turn implies

$$\frac{S(A, t)}{\sigma_{n-1}} \leq \left( \frac{\text{diam}(A) + 2t}{2} \right)^{n-1}.$$

So, the last inequality, along with (2.4), gives

$$\begin{aligned}
& \frac{\text{pcap}(r\mathbb{B}^n, A)}{\sigma_{n-1}} \\
& \leq \left( \int_0^\tau \left( \frac{\text{diam}(A) + 2t}{2} \right)^{\frac{n-1}{1-p}} dt \right)^{1-p} \\
& = \left( \left( \frac{1-p}{n-p} \right) \left( \left( \frac{\text{diam}(A)}{2} + \tau \right)^{\frac{n-p}{1-p}} - \left( \frac{\text{diam}(A)}{2} \right)^{\frac{n-p}{1-p}} \right) \right)^{1-p} \\
& \rightarrow \left( \left( \frac{p-1}{n-p} \right) \left( \frac{\text{diam}(A)}{2} \right)^{\frac{n-p}{1-p}} \right)^{1-p} \quad \text{as } \tau \rightarrow \infty.
\end{aligned}$$

As a result, we get

$$\frac{\text{pcap}(A)}{\sigma_{n-1}} = \lim_{r \rightarrow \infty} \frac{\text{pcap}(r\mathbb{B}^n, A)}{\sigma_{n-1}} \leq \left( \frac{p-1}{n-p} \right)^{1-p} \left( \frac{\text{diam}(A)}{2} \right)^{n-p}$$

whence reaching the inequality of (2.2).  $\square$

**2.2. Mean radius.** For  $A \in C^n$ , denote by (cf. [57, 1.7])

$$h_A(x) = \sup_{y \in A} x \cdot y \quad \& \quad b(A) = \frac{2}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} h_A d\theta$$

the support function and the mean width of  $A$  (with  $d\theta$  being the standard area measure on  $\mathbb{S}^{n-1}$ ) respectively, and then write  $b(A)/2$  for the mean radius of  $A$  according to [54]. Clearly,

$$\frac{b(A)}{2} \leq \frac{\text{diam}(A)}{2}$$

with equality if  $A$  is a ball. Interestingly, the Uryasohn inequality (cf. [57, (6.25)])

$$(2.5) \quad \left( \frac{\mathcal{L}^n(A)}{\omega_n} \right)^{\frac{1}{n}} \leq \frac{b(A)}{2}$$

holds with equality if  $A$  is a ball. Even more interestingly, the forthcoming lemma reveals that (2.5) can be further improved.

**Lemma 2.2.**

(i) If  $A \in C^n$  and  $p = n - 1$ , then

$$(2.6) \quad \left( \left( \frac{p-1}{n-p} \right)^{p-1} \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right) \right)^{\frac{1}{n-p}} \leq \frac{b(A)}{2}$$

with equality if  $A$  is a ball.

(ii) If  $A \in C^n$ , then

$$(2.7) \quad \left( \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}} \leq \frac{b(A)}{2}$$

with equality if  $A$  is a ball.

*Proof.* Since (2.7) can be seen from Chakerian's [7, (25)], it is enough to verify (2.6). Note that

$$(2.8) \quad \frac{|x|b(A)}{2} = \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} h_A(|x|\theta) d\theta.$$

is valid for any given  $x \in \mathbb{R}^n$ , and importantly, an extension of [2, Example 7.4] to  $A \in C^n$  tells us that the right side of (2.8) can be approximated by  $\sum_{j=1}^m h_A(|x|\theta_j)\lambda_j$  which is the support function of  $\sum_{j=1}^m \lambda_j R_j(A)$ , where

$$\begin{cases} \lambda_j \in (0, 1); \\ \sum_{j=1}^m \lambda_j = 1; \end{cases}$$

and  $R_j(A)$  is an appropriate rotation of  $A$  associated to  $\theta_j$ . Therefore, by employing Colesanti-Salani's [14, Theorem 1] and by induction, we can readily obtain that if  $p = n - 1$  then

$$(2.9) \quad \text{pcap}(A) = \sum_{j=1}^m \lambda_j \text{pcap}(A) = \sum_{j=1}^m \lambda_j \text{pcap}(R_j(A)) \leq \text{pcap}\left(\sum_{j=1}^m \lambda_j R_j(A)\right).$$

Here the rotation-invariance of  $\text{pcap}(\cdot)$  has been used; see e.g. [20, Page 151]. Note also that the left side of (2.8) is the support function of a ball of radius  $b(A)/2$ . So, a combination of the above approximation, the correspondence between a support function and a convex set, (2.9) and the well-known formula

$$(2.10) \quad \text{pcap}(r\mathbb{B}^n) = \sigma_{n-1} \left( \frac{p-1}{n-p} \right)^{1-p} r^{n-p},$$

derives the left inequality of (2.6).  $\square$

**2.3.  $p - 1$  integral mean curvature radius.** We should point out that if  $p = n - 1 = 2$  then (2.6) is just Pólya's inequality [54, (5)] – here the fact that for a  $C^2$  body  $A \in \mathbb{K}^3$  the mean radius  $b(A)/2$  is equal to  $(4\pi)^{-1}$  times the surface integral of the mean curvature has been used. To see this more transparently, let us recall that for a convex set  $A$  with its boundary  $\partial A$  being  $C^2$  hypersurface,

$$m_j(A, x) = \begin{cases} 1 & \text{for } j = 0; \\ \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \dots < i_j \leq n-1} \kappa_{i_1}(x) \cdots \kappa_{i_j}(x) & \text{for } j = 1, \dots, n-1, \end{cases}$$



is the  $j$ -th mean curvature at  $x \in \partial A$ , where  $\kappa_1(x), \dots, \kappa_{n-1}(x)$  are the principal curvatures of  $\partial A$  at the point  $x$ . Note that (see, e.g. [22])

$$\begin{cases} m_1(A, x) = H(\partial A, x) = \text{mean curvature of } \partial A \text{ at } x; \\ m_j(A, x) \leq (H(\partial A, x))^j \quad \text{for } j = 1, \dots, n-1; \\ m_{n-1}(A, x) = G(\partial A, x) = \text{Gauss curvature of } \partial A \text{ at } x. \end{cases}$$

Such a higher order mean curvature  $m_j(A, \cdot)$  is used to produce the so-called  $j$ -th integral mean curvature of  $\partial A$ :

$$M_j(A) = \int_{\partial A} m_j(A, \cdot) d\mathcal{H}^{n-1}(\cdot).$$

Clearly, we have

$$\begin{cases} M_0 = \mathcal{H}^{n-1}(\partial A); \\ M_1 = \int_{\partial A} H(\partial A, \cdot) d\mathcal{H}^{n-1}(\cdot); \\ M_{n-2} = \sigma_{n-1} b(K)/2. \end{cases}$$

Moreover, if  $\nu(x)$  is the outer unit normal vector then (cf. [48])

$$M_0 = \int_{\partial A} x \cdot \nu(x) H(\partial A, x) d\mathcal{H}^{n-1}(x);$$

if  $n = 2$  then the Gauss-Bonnet formula gives  $M_1(A) = 2\pi$ ; and if  $p = n - 1 = 2$  then (2.6) reduces to the above-mentioned Pólya's inequality.

According to [56, (13.43)], the foregoing  $S(A, t)$  has the following decomposition

$$S(A, t) = \sum_{j=0}^{n-1} \binom{n-1}{j} M_j(A) t^j.$$

This formula is brought into (2.4) to deduce

$$(2.11) \quad \text{pcap}(A) \leq \left( \int_0^\infty \left( \int_{\partial A} (1 + tH(\partial A, \cdot))^{n-1} d\mathcal{H}^{n-1}(\cdot) \right)^{\frac{1}{1-p}} dt \right)^{1-p}$$

with equality if  $A$  is a ball. The inequality (2.11) will be complemented through the forthcoming lemma which not only extends Freire-Schwartz's [22, Theorem 2] (and Pólya's inequalities [54, (3)-(4)] for  $n = 3$ ) from  $p = 2$  to  $p \in [2, n)$ , but also induces Willmore's inequality (cf. [55, Page 87] or [8] for immersed hypersurfaces in  $\mathbb{R}^n$ ):

$$\sigma_{n-1} \leq \int_{\partial A} (H(\partial A, \cdot))^{n-1} d\mathcal{H}^{n-1}(\cdot)$$

through letting  $p \rightarrow n$  in (2.13) whose special case  $p = 2$  is essentially the Huisken's result presented in [28, Theorem 6].

**Lemma 2.3.** *Given  $p \in [2, n)$ , let  $A \subset \mathbb{R}^n$  be a connected compact set with  $C^2$  boundary  $\partial A$ .*

(i) *If  $H(\partial A, \cdot) \geq 0$ , then*

$$(2.12) \quad \left( \left( \frac{p-1}{n-p} \right)^{p-1} \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right) \right)^{\frac{1}{n-p}} \leq \left( \frac{1}{\sigma_{n-1}} \int_{\partial A} (H(\partial A, \cdot))^{p-1} d\mathcal{H}^{n-1}(\cdot) \right)^{\frac{1}{n-p}}$$

with equality if  $A$  is a ball.

(ii) *If  $H(\partial A, \cdot) \geq 0$  and  $\partial A$  is outer-minimizing, i.e.,  $K \supseteq A \Rightarrow \mathcal{H}^{n-1}(\partial K) \geq \mathcal{H}^{n-1}(\partial A)$ , then*

$$(2.13) \quad \left( \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}} \leq \left( \frac{1}{\sigma_{n-1}} \int_{\partial A} (H(\partial A, \cdot))^{p-1} d\mathcal{H}^{n-1}(\cdot) \right)^{\frac{1}{n-p}}$$

with equality if  $A$  is a ball.

*Proof.* Geometrically speaking, the right hand side of (2.12)/(2.13) is said to be the  $p-1$  integral mean curvature radius of  $A$ . Obviously, (2.12)/(2.13) and (1.5)/(1.6) are combined to deduce the following volume-integral-mean-curvature inequality

$$\left( \frac{\mathcal{L}^n(A)}{\omega_n} \right)^{\frac{1}{n}} \leq \left( \frac{1}{\sigma_{n-1}} \int_{\partial A} (H(\partial A, \cdot))^{p-1} d\mathcal{H}^{n-1}(\cdot) \right)^{\frac{1}{n-p}}$$

with equality if  $A$  is a ball.

The equality cases of (2.12) and (2.13) are trivial. So, it remains to check their inequalities. To do so, we will write  $(\Sigma_t)_{t \geq 0}$  for the level sets of the function induced by the weak solution  $\phi$  to Huisken-Ilmanen's Inverse Mean Curvature Flow (IMCF) (cf. [34, 35]) starting at  $\partial A$ :  $\frac{\partial \phi}{\partial t} = H^{-1} \nu$  – here  $\nu$  is the outer unit normal vector and the level set formulation of this flow is decided by the Dirichlet problem:

$$\begin{cases} \operatorname{div}(|\nabla u|^{-1} \nabla u) = |\nabla u| & \text{in } \mathbb{R}^n \setminus A; \\ u = 0 & \text{on } \partial A \quad \& \quad u(x) \rightarrow \infty \quad \text{as } |x| \rightarrow \infty, \end{cases}$$

whose proper weak solution (cf. [49, 50]) can be obtained via letting  $p \rightarrow 1$  in  $v = \exp(u/(1-p))$  coupled with the boundary value problem (see also (1.2)):

$$\begin{cases} \operatorname{div}(|\nabla v|^{p-2} \nabla v) = 0 & \text{in } \mathbb{R}^n \setminus A; \\ v = 1 & \text{on } \partial A \quad \& \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{cases}$$

The following fundamental results (a)-(b)-(c)-(d) on IMCF are due to Huisken-Ilmanen (cf. [34, 35] and [4, 22]):

(a) There is a proper, locally Lipschitz function  $\phi$  such that:  $\phi \geq 0$  in  $\mathbb{R}^n \setminus A^\circ$ ;  $\phi = 0$  on  $\partial A$ ; for each  $t > 0$ ,

$$\Sigma_t = \partial\{x \in \mathbb{R}^n \setminus A^\circ : \phi(x) \geq t\} \quad \& \quad \Sigma'_t = \partial\{x \in \mathbb{R}^n \setminus A^\circ : \phi(x) > t\}$$

define increasing families of  $C^{1,\alpha}$  hypersurfaces.

(b)  $\Sigma_t$  (resp.  $\Sigma'_t$ ) minimize (resp. strictly minimize) area among surfaces homologous to  $\Sigma_t$  in  $\{x \in \mathbb{R}^n \setminus A^\circ : \phi(x) \geq t\}$ ;

$$\Sigma' = \partial\{x \in \mathbb{R}^n \setminus A^\circ : \phi(x) > 0\}$$

strictly minimizes area among hypersurfaces homologous to  $\Sigma = \partial A$  in  $\mathbb{R}^n \setminus A^\circ$ .

(c) For almost all  $t > 0$ , the weak mean curvature of  $\Sigma_t$  is defined and equal to  $|\nabla\phi|/(n-1)$  that is positive almost everywhere on  $\Sigma_t$ .

(d) For each  $t > 0$ , one has:

$$\mathcal{H}^{n-1}(\Sigma_t) = e^t \mathcal{H}^{n-1}(\Sigma');$$

and

$$\mathcal{H}^{n-1}(\Sigma_t) = e^t \mathcal{H}^{n-1}(\partial A)$$

if  $\partial A$  is outer-minimizing, i.e.,  $\partial A$  minimizes area among all surfaces homologous to  $\partial A$  in  $\mathbb{R}^n \setminus A^\circ$ .

According to (1.1) and the definition of  $\text{pcap}(\cdot)$ , we have

$$\text{pcap}(A) = \text{pcap}(\partial A) \leq \inf_f \int_{\mathbb{R}^n \setminus A^\circ} |\nabla f|^p d\mathcal{L}^n$$

where the infimum is taken over all functions  $f = g \circ \psi$  that have the above-described level hypersurfaces  $(\Sigma_t)_{t \geq 0}$  and enjoy the property that  $g$  is a one-variable function with  $g(0) = 0$  and  $g(\infty) = 1$  and  $\psi$  is a nonnegative function on  $\mathbb{R}^n \setminus A^\circ$  with  $\psi|_{\partial A} = 0$  and  $\lim_{|x| \rightarrow \infty} \psi(x) = \infty$ . Note that the classical co-area formula yields

$$\int_{\mathbb{R}^n \setminus A^\circ} |\nabla f|^p d\mathcal{L}^n = \int_0^\infty |g'(t)|^p \left( \int_{\Sigma_t} |\nabla \psi|^{p-1} d\mathcal{H}^{n-1} \right) dt.$$

So, upon choosing

$$\begin{cases} \psi = \phi; \\ g(t) = V_p(t) = \frac{\int_0^t (U_p(s))^{\frac{1}{1-p}} ds}{\int_0^\infty (U_p(s))^{\frac{1}{1-p}} ds}; \\ U_p(t) = \frac{1}{\sigma_{n-1}} \int_{\Sigma_t} |\nabla \phi|^{p-1} d\mathcal{H}^{n-1}, \end{cases}$$

we can achieve

$$\frac{\text{pcap}(\partial A)}{\sigma_{n-1}} \leq \int_0^\infty U_p(t) \left| \frac{d}{dt} V_p(t) \right|^p dt,$$

whence finding

$$(2.14) \quad \frac{\text{pcap}(A)}{\sigma_{n-1}} \leq \left( \int_0^\infty (U_p(t))^{\frac{1}{1-p}} dt \right)^{1-p}.$$

Next, let us estimate the growth of  $U_p(\cdot)$ . In fact, utilizing [35, Lemma 1.2, (ii)&(v)], an integration-by-part, the inequality

$$H_t^2 - (n-1)|\Pi_t|^2 \leq 0$$

with  $0 < H_t$  and  $\Pi_t$  being the mean curvature and the second fundamental form on  $\Sigma_t$  respectively, the assumption  $p \in [2, n)$ , and the property (c) above, we get

$$\begin{aligned} & \frac{d}{dt} U_p(t) \\ &= \frac{d}{dt} \left( \frac{(n-1)^{p-1}}{\sigma_{n-1}} \int_{\Sigma_t} H_t^{p-1} d\mathcal{H}^{n-1} \right) \\ &= \frac{(n-1)^{p-1}}{\sigma_{n-1}} \int_{\Sigma_t} \left( (p-1)H_t^{p-2} \left( \frac{d}{dt} H_t \right) + H_t^{p-1} \right) d\mathcal{H}^{n-1} \\ &= \frac{(n-1)^{p-1}}{\sigma_{n-1}} \int_{\Sigma_t} \left( 1 - (p-1) \left( \frac{|\Pi_t|}{H_t} \right)^2 - (p-2)|\nabla H_t^{-1}|^2 \right) H_t^{p-1} d\mathcal{H}^{n-1} \\ &\leq \frac{n-p}{(n-1)\sigma_{n-1}} \int_{\Sigma_t} |\nabla \phi|^{p-1} d\mathcal{H}^{n-1} \\ &= \left( \frac{n-p}{n-1} \right) U_p(t). \end{aligned}$$

Here, the author thanks Professor Guofang Wang for pointing out that  $\Delta H_t^{-1}$  should appear in the above argument.

The last estimate in turn implies

$$(2.15) \quad U_p(t) \leq U_p(0) \exp \left( t \left( \frac{n-p}{n-1} \right) \right).$$

Using (2.14)-(2.15) we find

$$\frac{\text{pcap}(A)}{\sigma_{n-1}} \leq U_p(0) \left( \frac{(n-1)(p-1)}{n-p} \right)^{1-p}$$

whence reaching (2.12) via (c).

Finally, in order to check (2.13), we apply (d) and the above-established differential inequality

$$\frac{d}{dt} U_p(t) \leq \left( \frac{n-p}{n-1} \right) U_p(t)$$

to discover that

$$t \mapsto \Phi_p(t) = (\mathcal{H}^{n-1}(\Sigma_t))^{\frac{p-n}{n-1}} \int_{\Sigma_t} H_t^{p-1} d\mathcal{H}^{n-1}$$

is a decreasing function. But, since  $\Sigma_t$  tends to a round sphere as  $t \rightarrow \infty$ , one concludes

$$\Phi_p(\infty) = \lim_{t \rightarrow \infty} \Phi_p(t) = \sigma_{n-1}^{\frac{p-1}{n-1}}.$$

Therefore,

$$(\mathcal{H}^{n-1}(\partial A))^{\frac{p-n}{n-1}} \int_{\partial A} (H(\partial A, \cdot))^{p-1} d\mathcal{H}^{n-1}(\cdot) = \Phi_p(0) \geq \Phi_p(\infty) = \sigma_{n-1}^{\frac{p-1}{n-1}},$$

and consequently, (2.13) follows.  $\square$

**2.4. Variational  $p$  capacity radius vs surface radius.** In his paper [54], Pólya conjectured that of all members in  $C^3$ , with a given surface area, the round ball has the minimal electrostatic capacity  $2cap(\cdot)$ . While this conjecture has not yet been proved or disproved, the following Lemma 2.4 confirms partially the conjecture.

**Lemma 2.4.** *Let  $p \in (1, n)$ .*

(i) *If there is a constant  $\alpha > 0$  such that  $A \subset \mathbb{R}^n$  is  $\alpha$ -convex, i.e., for any  $x \in \partial A$  there exists a closed ball  $B$  with radius  $\alpha^{-1}$  such that  $x \in \partial B$  and  $A \subseteq B$ , then*

$$(2.16) \quad \left( \frac{p-1}{(n-p)\alpha} \right)^{p-1} \left( \frac{pcap(A)}{\sigma_{n-1}} \right) \geq \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}}$$

with equality when and only when  $A$  is a ball of radius  $\alpha^{-1}$ .

(ii) *If  $A \subset \mathbb{R}^n$  is a connected compact set with  $C^2$  boundary  $\partial A$  and there is a constant  $\beta > 0$  such that  $0 \leq H(\partial A, \cdot) \leq \beta$ , then*

$$(2.17) \quad \left( \frac{p-1}{(n-p)\beta} \right)^{p-1} \left( \frac{pcap(A)}{\sigma_{n-1}} \right) \leq \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}}$$

with equality when and only when  $A$  is a ball of radius  $\beta^{-1}$ .

*Proof.* (i) To prove (2.16), let us keep in mind the fact that if  $\partial A$  is of  $C^2$  then  $A$  is  $\alpha$ -convex if and only if each principal curvature  $\kappa_j$  of  $\partial A$  is not less than  $\alpha$ , i.e.,  $\kappa_j \geq \alpha$ .

Following the argument for Hurtado-Palmer-Ritoré's [36, Theorem 4.5] which is just the case  $p = 2$  of (2.16) we set

$$v(x) = \phi(d(x, A)) \quad \& \quad \phi(t) = (1 + \alpha t)^{\frac{p-n}{p-1}}.$$

Then  $v$  is of  $C^{1,1}$  in  $\mathbb{R}^n \setminus A$ . Given  $t \in (0, \infty)$ . If  $x \in \mathbb{R}^n \setminus A$  is such a point that  $d(x, A)$  is twice differentiable along the line minimizing  $d(x, A)$  and if

$$A_t = \{y \in \mathbb{R}^n : \text{dist}(y, A) \leq t\},$$

then on this line one utilizes (1.4) to derive

$$\text{div}(|\nabla v|^{p-2} \nabla v) = |\phi'(d(x, A))|^{p-2} \left( (n-1)H_t(x)\phi'(d(x, A)) + (p-1)\phi''(d(x, A)) \right)$$

where  $H_t$  stands for the mean curvature of the hypersurface  $\partial A_t$  which is parallel to  $\partial A$ . Note that  $A_t$  is  $(t + \alpha^{-1})^{-1}$ -convex. So, one has

$$(2.18) \quad H_t \geq \alpha/(1 + \alpha t)$$

at the regular points in  $\partial A_t$ . Recall that  $u = u_A$  is the  $p$ -equilibrium potential. A simple calculation gives

$$\phi'(t) = \alpha \left( \frac{p-n}{p-1} \right) (1 + t\alpha)^{\frac{1-n}{p-1}} \leq 0.$$

This, along with (2.18) and a simple computation, shows that

$$\begin{aligned} & \text{div}(|\nabla v|^{p-2} \nabla v) \\ &= |\phi'(d(x, A))|^{p-2} \alpha(n-1) \left( \frac{p-n}{p-1} \right) (1 + d(x, A)\alpha)^{\frac{1-n}{p-1}-1} \left( (1 + \alpha d(x, A))H_t - \alpha \right) \\ &\leq 0 = \text{div}(|\nabla u|^{p-2} \nabla u) \end{aligned}$$

holds whenever  $x \mapsto d(x, A)$  is of  $C^2$ .

Next, we prove that  $v \geq u$  holds in  $\mathbb{R}^n \setminus A$ . For the above given  $t > 0$  let  $u_t$  and  $\phi_t$  be the  $p$ -equilibrium potentials of the rings

$$(A_t, A) \quad \& \quad ((t + \alpha^{-1})\mathbb{B}^n, \alpha^{-1}\overline{\mathbb{B}^n})$$

respectively (cf. [41]), as well as, set  $v_t = \phi_t(d(x, A))$ . Then the last div-estimate, plus an integration-by-part argument, implies that

$$\text{div}(|\nabla v_t|^{p-2} \nabla v_t) \leq \text{div}(|\nabla u_t|^{p-2} \nabla u_t) \quad \text{in } A_t \setminus A$$

is valid in the distributional sense. Now, from the weak comparison principle for  $p$ -Laplacian (see e.g. [61]) it follows that  $v_t \geq u_t$  holds in  $A_t \setminus A$ , and so that  $v \geq u$  is valid in  $\mathbb{R}^n \setminus A$  via letting  $t \rightarrow \infty$ .

Note also that  $\nabla u$  and  $\nabla v$  have non-tangential limit  $\mathcal{H}^{n-1}$ -almost everywhere on  $\partial A$ . So, if  $x \in \partial A$ , then  $\nabla u$  and  $\nabla v$  can be defined at  $x$ . Upon extending  $u$  and  $v$  continuously to  $x$  and  $B$  being an exterior ball to  $A$ , and utilizing

$$(2.19) \quad \begin{cases} \text{div}(|\nabla v|^{p-2} \nabla v) \leq \text{div}(|\nabla u|^{p-2} \nabla u) & \text{in } B; \\ u(x) = v(x) = 1 & \text{for } x \in \partial A; \\ v(x) \geq u(x) & \text{for } x \in \mathbb{R}^n \setminus A; \\ v - u & \text{continuous on } B \cup \partial B, \end{cases}$$

as well as taking the Hopf maximum principle into account, we get

$$(2.20) \quad |\nabla v(x)| \leq |\nabla u(x)| \quad \forall \quad x \in \partial A.$$

An application of (1.1) gives that

$$(2.21) \quad \begin{aligned} & \text{pcap}(A) \\ &= \int_{\partial A} |\nabla u|^{p-1} d\mathcal{H}^{n-1} \\ &\geq \int_{\partial A} |\nabla v|^{p-1} d\mathcal{H}^{n-1} \\ &= (-\phi'(0))^{p-1} \mathcal{H}^{n-1}(\partial A) \\ &= \left( \left( \frac{n-p}{p-1} \right) \alpha \right)^{p-1} \mathcal{H}^{n-1}(\partial A), \end{aligned}$$

namely, (2.16) holds.

Of course, if  $A$  is a ball with radius  $\alpha^{-1}$ , then equality of (2.16) trivially holds. Conversely, when equality of (2.16) is true, (2.21) is employed to derive that  $|\nabla u(x)| = |\nabla v(x)|$  holds for  $\mathcal{H}^{n-1}$ -almost every points  $x \in \partial A$ . Consequently,  $u = v$  holds on any exterior ball to  $A$  and therefore it still true in  $\mathbb{R}^n \setminus A$ . So, the level sets of  $u$  and  $v$  are the same. Thanks to  $u \in C^\infty(\mathbb{R}^n \setminus A)$  (cf. [14]), the level sets of  $u$  are  $C^\infty$  hypersurfaces. Since

$$|\nabla v(x)| = |\phi'(d(x, A))| |\nabla d(x, A)| \neq 0 \quad \forall \quad x \in \mathbb{R}^n \setminus A,$$

one has that  $|\nabla u| = |\nabla v|$  does not vanish. Consequently,

$$\begin{cases} H_t = \alpha/(1 + \alpha t); \\ \text{div}(|\nabla v|^{p-2} \nabla v) = \text{div}(|\nabla u|^{p-2} \nabla u). \end{cases}$$

This in turn derives that the principal curvatures of  $\partial A_t$  equal  $(t + \alpha^{-1})^{-1}$ , and so that  $(A_t)_{t>0}$  are concentric balls with radius  $\alpha^{-1} + t$ . Therefore,  $A$  is a ball of radius  $\alpha^{-1}$ .

(ii) Under the assumption that  $A \subset \mathbb{R}^n$  is a connected compact set with  $C^2$  boundary  $\partial A$  and  $0 \leq H(\partial A, \cdot) \leq \beta$  holds for a constant  $\beta > 0$ , we may apply (2.12) to derive that under  $p \in [2, n)$  one has

$$\frac{\left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right)}{\left( \frac{p-1}{n-p} \right)^{1-p}} \leq \frac{1}{\sigma_{n-1}} \int_{\partial A} \left( H(\partial A, \cdot) \right)^{p-1} d\mathcal{H}^{n-1}(\cdot) \leq \frac{\left( \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \right)}{\beta^{1-p}},$$

and thus (2.17).

Nevertheless, the general inequality (2.17) can be also verified by slightly modifying the above argument for (i). The key is the selection of the function pair  $(v, \phi)$  for (ii) - more precisely -

$$v(x) = \phi(d(x, A)) \quad \& \quad \phi(t) = (1 + \beta t)^{\frac{p-n}{p-1}}.$$

Under this choice,  $\alpha$ , (2.18), (2.20), and (2.21) will be replaced by

$$\begin{cases} \beta, \\ H_t \leq \beta/(1 + \beta t), \\ \begin{cases} \operatorname{div}(|\nabla v|^{p-2} \nabla v) \geq \operatorname{div}(|\nabla u|^{p-2} \nabla u) & \text{in } B; \\ u(x) = v(x) = 1 & \text{for } x \in \partial A; \\ v(x) \leq u(x) & \text{for } x \in \mathbb{R}^n \setminus A; \\ v - u & \text{continuous on } B \cup \partial B, \\ |\nabla v(x)| \geq |\nabla u(x)| & \forall x \in \partial A, \end{cases} \end{cases}$$

and

$$\begin{aligned} \operatorname{pcap}(A) &= \int_{\partial A} |\nabla u|^{p-1} d\mathcal{H}^{n-1} \\ &\leq \int_{\partial A} |\nabla v|^{p-1} d\mathcal{H}^{n-1} \\ &= \left( \frac{n-p}{p-1} \beta \right)^{p-1} \mathcal{H}^{n-1}(\partial A), \end{aligned}$$

as desired.

The argument for equality of (2.17) is similar to that for equality of (2.6) (but this time, just using the last estimation), and so left for the interested reader.  $\square$

**2.5. Graphic ADM mass radius.** Following [40] we consider the so-called graphic ADM mass. For  $f(x) = f(x_1, \dots, x_n)$  and  $i, j, k = 1, 2, \dots, n$  we write

$$\begin{cases} f_i = \frac{\partial f}{\partial x_i}; \\ f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}; \\ f_{ijk} = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}; \\ \delta_{ij} = 0 \text{ or } 1 \text{ as } i \neq j \text{ or } i = j. \end{cases}$$

Suppose  $U$  is a bounded open set in  $\mathbb{R}^n$  with boundary  $\partial U$ . We say that a smooth function  $f : \mathbb{R}^n \setminus U \mapsto \mathbb{R}$  is asymptotically flat provided there is a constant  $\gamma > (n-2)/2$  such that

$$|f_i(x)| + |x| |f_{ij}(x)| + |x|^2 |f_{ijk}(x)| = O(|x|^{-\gamma/2}) \quad \text{as } |x| \rightarrow \infty.$$

Now, given such a smooth asymptotically flat function, the graph of  $f$ , denoted by

$$(\mathbb{R}^n \setminus U, \delta + df \otimes df) = (\mathbb{R}^n \setminus U, (\delta_{ij} + f_i f_j)),$$



is a complete Riemannian manifold. And then, the ADM (named after three physicists: Arnowitt, Deser and Misner) mass of this graph is determined by

$$m_{ADM}(\mathbb{R}^n \setminus U, \delta + df \otimes df) = \lim_{r \rightarrow \infty} \int_{S_r} \sum_{i,j=1}^n \frac{(f_{ii}f_j - f_{ij}f_i)x_j|x|^{-1}}{2(n-1)\sigma_{n-1}(1 + |\nabla f|^2)} d\sigma,$$

where  $S_r$  is the coordinate sphere of radius  $r$  and  $d\sigma$  is the area element of  $S_r$ . It is perhaps appropriate to point out that under  $n \geq 3$  this definition of the ADM mass coincides the definition of the original ADM mass of an asymptotically flat manifold; see also [40] for a brief review on the Riemannian positive mass theorem (cf. Schoen-Yau ([58, 59]) and Witten [70]) and its strengthening - the Riemannian Penrose inequality for area outer minimizing horizon (cf. Huisken-Ilmanen [34] and Bray [3]).

**Lemma 2.5.** *Let  $A \subset \mathbb{R}^n$  be a connected compact set with  $C^2$  boundary  $\partial A$  and  $H(\partial A, \cdot) \geq 0$ . Suppose  $f : \mathbb{R}^n \setminus A^\circ \mapsto \mathbb{R}$  is a smooth asymptotically flat function such that  $f(\partial A)$  is in a level set of  $f$ ,  $\lim_{x \rightarrow \partial A} |\nabla f(x)| = \infty$ , and the scalar curvature of  $(\mathbb{R}^n \setminus A^\circ, \delta + df \otimes df)$  is nonnegative.*

(i) *If  $p = 2 < n$ , then*

$$(2.22) \quad \left( \left( \frac{p-1}{n-p} \right)^{p-1} \frac{pcap(A)}{\sigma_{n-1}} \right)^{\frac{1}{n-p}} \leq (2m_{ADM}(\mathbb{R}^n \setminus A^\circ, \delta + df \otimes df))^{\frac{1}{n-2}}.$$

(ii) *If  $n \geq 3$  and  $\partial A$  is outer-minimizing, then*

$$(2.23) \quad \left( \frac{\mathcal{H}^{n-1}(\partial A)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}} \leq (2m_{ADM}(\mathbb{R}^n \setminus A^\circ, \delta + df \otimes df))^{\frac{1}{n-2}}.$$

*Proof.* Naturally, the right hand side of (2.22)/(2.23) is called the graphic ADMS mass radius. An application of both (2.22)/(2.23) and (1.5)/(1.6) gives the following volume-mass inequality

$$\left( \frac{\mathcal{L}^n(A)}{\omega_n} \right)^{\frac{1}{n}} \leq (2m_{ADM}(\mathbb{R}^n \setminus A^\circ, \delta + df \otimes df))^{\frac{1}{n-2}};$$

see also [60] for an analogous inequality for the conformally flat manifolds.

As the Penrose inequality for graphs with convex boundaries, (2.23) for  $A \in C^n$  comes from Lam's [40, Remark 8]. Since [40, Lemma 12] can be replaced by (2.13) under  $p = 2$  and  $\partial A$  being outer-minimizing, (2.23) is valid for the case described in Lemma 2.5. Thus, it remains to verify (2.22).

Note that Lam's [40, Theorem 6] actually says

$$\begin{aligned}
 (2.24) \quad & 2m_{ADM}(\mathbb{R}^n \setminus A^\circ, \delta + df \otimes df) \\
 &= \frac{1}{\sigma_{n-1}} \int_{\partial A} H(\partial A, \cdot) d\mathcal{H}^{n-1}(\cdot) \\
 &+ \frac{1}{(n-1)n\omega_n} \int_{\mathbb{R}^n \setminus A^\circ} R_f(\cdot) d\mathcal{L}^n(\cdot),
 \end{aligned}$$

where

$$R_f = \sum_{j=1}^n \frac{\partial}{\partial x_j} \sum_{i=1}^n \left( \frac{f_{ii}f_j - f_{ij}f_i}{1 + |\nabla f|^2} \right)$$

is the scalar curvature of the graph  $(\mathbb{R}^n \setminus A^\circ, \delta + df \otimes df)$  of  $f$ ; see also [40, Lemma 10] or [33, Proposition 5.4]. Thus, (2.24), which may be regarded as the Gauss-Bonnet like formula for the graphic ADM mass, along with  $R_f \geq 0$ , and (2.12), implies that if  $p = 2 < n$  then

$$\begin{aligned}
 & \left( \left( \frac{p-1}{n-p} \right)^{p-1} \left( \frac{\text{pcap}(A)}{\sigma_{n-1}} \right) \right)^{\frac{1}{n-p}} \\
 & \leq \left( \frac{1}{\sigma_{n-1}} \int_{\partial A} (H(\partial A, \cdot))^{p-1} d\mathcal{H}^{n-1}(\cdot) \right)^{\frac{1}{n-p}} \\
 & \leq \left( 2m_{ADM}(\mathbb{R}^n \setminus A^\circ, \delta + df \otimes df) \right)^{\frac{1}{n-p}}.
 \end{aligned}$$

In other words, (2.22) holds.  $\square$

### 3. PROOFS OF THEOREM AND ITS COROLLARY

We are ready to prove Theorem 1.1 and its Corollary 1.2.

*Proof of Theorem 1.1.* (i) Owing to  $h \in L^1(\mathbb{R}^n)$ , we get

$$F_{\text{pcap}}(A) \leq \|h\|_{L^1(\mathbb{R}^n)} - \text{pcap}(A) \quad \forall \quad A \in C^n.$$

Observe that if  $(B_j)_{j \geq 1}$  is a sequence of closed balls converging to a point then  $(F_{\text{pcap}}(B_j))_{j \geq 1}$  tends to 0. Thus,  $\sup_{A \in C^n} F_{\text{pcap}}(A) \geq 0$ . As a consequence, if  $F_{\text{pcap}}(\cdot)$  attains its supremum at  $A_0 \in C^n$  then there must be

$$F_{\text{pcap}}(A_0) = \sup_{A \in C^n} F_{\text{pcap}}(A) \geq 0.$$

On the other hand, suppose there exists  $A \in C^n$  so that  $F_{\text{pcap}}(A) \geq 0$ . Then  $\sup_{K \in C^n} F_{\text{pcap}}(K) \geq 0$ . If  $(A_j)_{j \geq 1}$  is a sequence of maximizers for  $F_{\text{pcap}}(\cdot)$  with  $F_{\text{pcap}}(A_j) > 0$  and the inradius of  $A_j$  having a uniform lower bound  $r_0 > 0$  (if, otherwise,  $A_j$  converges to a set of single point  $\{a_0\}$ , then

$F_{\text{pcap}}(A_j)$  tends to 0 and hence  $\{a_0\} \in C^n$  is a maximizer). Using this, (2.10) and (2.2) of Lemma 2.1 we obtain

$$\begin{aligned} r_0 &= \left( \left( \frac{p-1}{n-p} \right)^{p-1} \left( \frac{\text{pcap}(r_0 \mathbb{B}^n)}{\sigma_{n-1}} \right) \right)^{\frac{1}{n-p}} \\ (3.1) \quad &\leq \left( \left( \frac{p-1}{n-p} \right)^{p-1} \left( \frac{\text{pcap}(A_j)}{\sigma_{n-1}} \right) \right)^{\frac{1}{n-p}} \\ &\leq 2^{-1} \text{diam}(A_j). \end{aligned}$$

Because

$$(3.2) \quad F_{\text{pcap}}(A_j) \leq \|h\|_{L^1(\mathbb{R}^n)} - \sigma_{n-1} \left( \frac{p-1}{n-p} \right)^{1-p} \left( \frac{\mathcal{L}^n(A_j)}{\omega_n} \right)^{\frac{n-p}{n}},$$

one concludes that if  $(\text{diam}(A_j))_{j \geq 1}$  is not bounded, then (3.1) and the definition of  $r_0$  are employed to derive that  $(\mathcal{L}^n(A_j))_{j \geq 1}$  is not bounded, and so  $(F_{\text{pcap}}(A_j))_{j \geq 1}$  has a subsequence which goes to negative infinity. However, each  $F_{\text{pcap}}(A_j)$  is assumed to be positive. Thus,  $(\text{diam}(A_j))_{j \geq 1}$  has a uniform upper bound. Now, by the well-known Blaschke selection principle (see e.g. [57, Theorem 1.8.6]), we can choose a subsequence of  $(A_j)_{j \geq 1}$  that converges to  $A_0 \in \mathbb{K}^n$ . Since  $\text{pcap}(\cdot)$  is continuous (cf. [47, Pages 142-143]) and  $h \in C(\mathbb{R}^n)$  (i.e.,  $h$  is continuous in  $\mathbb{R}^n$ ),  $F_{\text{pcap}}(\cdot)$  is continuous, and so,  $A_0$  is a maximizer of  $F_{\text{pcap}}(\cdot)$ .

(ii) For  $A, B \in \mathbb{K}^n$  and  $t \in (0, 1)$  let  $C_t = A + tB$ . Then

$$C_t \in \mathbb{K}^n \quad \& \quad h_{C_t} = h_A + t h_B.$$

Using Tso's variational formula for  $\int_A h d\mathcal{L}^n$  in [66, (4)] and the variational formula for  $\text{pcap}(\cdot)$  in [13, Theorem 1.1] (see also [37, Corollary 3.16] or [38, Theorem 2.5] for  $2\text{cap}(\cdot)$ ), we obtain

$$(3.3) \quad \frac{d}{dt} F_{\text{pcap}}(C_t) \Big|_{t=0} = \int_{\partial A} h_B(\mathbf{g}) h d\mathcal{H}^{n-1} - \int_{\partial A} h_B(\mathbf{g}) (p-1) |\nabla u_A|^p d\mathcal{H}^{n-1}.$$

Obviously, if  $A$  is a maximizer of  $F_{\text{pcap}}(\cdot)$ , then it must be a critical point of  $F_{\text{pcap}}(C_t)$  and thus

$$\frac{d}{dt} F_{\text{pcap}}(C_t) \Big|_{t=0} = 0.$$

This and (3.3) derive

$$(3.4) \quad \int_{\partial A} h_B(\mathbf{g}) (p-1) |\nabla u_A|^p d\mathcal{H}^{n-1} = \int_{\partial A} h_B(\mathbf{g}) h d\mathcal{H}^{n-1}.$$

A combined application of (3.4) and [57, Lemmas 1.7.9 & 1.8.10] gives that

$$\begin{aligned}
& \int_{\mathbb{S}^{n-1}} \phi g_*((p-1)|\nabla u_A|^p d\mathcal{H}^{n-1}) \\
&= \int_{\partial A} \phi(g)(p-1)|\nabla u_A|^p d\mathcal{H}^{n-1} \\
&= \int_{\partial A} \phi(g)h d\mathcal{H}^{n-1} \\
&= \int_{\mathbb{S}^{n-1}} \phi g_*(h d\mathcal{H}^{n-1})
\end{aligned}$$

holds for any  $\phi \in C(\mathbb{S}^{n-1})$ , and thereby reaching (1.3). Moreover, if  $\partial A$  is  $C^2$  strictly convex, then the Gauss map from  $\partial A$  to  $\mathbb{S}^{n-1}$  is a diffeomorphism, and hence (1.3) is equivalent to

$$(p-1)|\nabla u_A(x)|^p = h(x) \quad \forall \quad x \in \partial A.$$

(iii) Suppose  $h \in C^{k,\alpha}$  with  $k$  being a nonnegative integer. Since  $\partial A$  is of  $C^2$ , an application of [43, Theorem 1] (cf. [24, 18, 62, 67, 26, 51]) yields that  $u_A \in C^{1,\hat{\alpha}}(A)$  is valid for some  $\hat{\alpha} \in (0, 1)$ . The last equation and  $h \in C^{k,\alpha}(\mathbb{R}^n)$  with  $\alpha \in (0, 1)$  derive that

$$|\nabla u_A| = \left( \frac{h}{p-1} \right)^{\frac{1}{p}}$$

is of  $C^{k,\alpha}$ . Note that  $\partial A$  is  $C^2$  strictly convex. So, if  $\partial A$  is represented locally as  $y_n = \psi(x_1, \dots, x_{n-1})$ , then the map

$$(x_1, \dots, x_{n-1}) \mapsto \nabla u_A(x_1, \dots, x_{n-1}, \psi(x_1, \dots, x_{n-1}))$$

is of  $C^{k,\alpha}$ . Thus, a combination of the chain rule (or the implicit function theorem) and the estimate  $0 < \inf_{\partial A} h \leq \sup_{\partial A} h < \infty$  imply that  $\psi$  is of  $C^{1+k,\alpha}$ . This in turn implies that  $\partial A$  is of  $C^{1+k,\alpha}$ .  $\square$

*Proof of Corollary 1.2.* The argument for Corollary (i) is very similar to that for Theorem 1.1(i) except that (3.1) and (3.2) are replaced respectively by their endpoint ( $p = 1$ ) cases:

$$r_0 = \left( \frac{\mathcal{H}^{n-1}(\partial(r_0\mathbb{B}^n))}{\sigma_{n-1}} \right)^{\frac{1}{n-1}} \leq \left( \frac{\mathcal{H}^{n-1}(\partial A_j)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}} \leq 2^{-1} \text{diam}(A_j)$$

and

$$F_{\mathcal{H}^{n-1}}(A_j) \leq \|h\|_{L^1(\mathbb{R}^n)} - \sigma_{n-1} \left( \frac{\mathcal{L}^n(A_j)}{\omega_n} \right)^{\frac{n-1}{n}}.$$

To reach Corollary (ii), recall that for  $C_t = A + tB$  with  $A, B \in \mathbb{K}^n$  and  $t \in (0, 1)$  (in the proof of Theorem 1.1 (ii)) there exists a curvature measure

$\mu_{\mathcal{H}^{n-1},A}$  on  $\mathbb{S}^{n-1}$  such that

$$\begin{cases} \mathcal{H}^{n-1}(\partial A) = (n-1)^{-1} \int_{\mathbb{S}^{n-1}} h_A d\mu_{\mathcal{H}^{n-1},A}; \\ \left. \frac{d}{dt} \mathcal{H}^{n-1}(\partial C_t) \right|_{t=0} = \int_{\mathbb{S}^{n-1}} h_B d\mu_{\mathcal{H}^{n-1},A}. \end{cases}$$

Since  $A$  is a maximizer of  $F_{\mathcal{H}^{n-1}}(\cdot)$ , it is a critical point of  $F_{\mathcal{H}^{n-1}}(\cdot)$ , and consequently,

$$\left. \frac{d}{dt} F_{\mathcal{H}^{n-1}}(C_t) \right|_{t=0} = 0,$$

whence yielding (1.7) via

$$d\mu_{\mathcal{H}^{n-1},A} = g_*(h d\mathcal{H}^{n-1}).$$

Furthermore, if  $\partial A$  is  $C^2$  strictly convex, then the Gauss map  $g : \partial A \mapsto \mathbb{S}^{n-1}$  is a diffeomorphic transformation, and hence (1.7) reduces to the mean curvature equation

$$h(x) = H(\partial A, x) \quad \forall \quad x \in \partial A$$

through using the variational formula for  $\mathcal{H}^{n-1}$  (see e.g. [15, 12, 10, 11])

$$\left. \frac{d}{dt} \mathcal{H}^{n-1}(\partial C_t) \right|_{t=0} = \int_{\partial A} h_B(g) H(\partial A, \cdot) d\mathcal{H}^{n-1}(\cdot).$$

To validate Corollary (iii), note once again that under  $\partial A$  being  $C^2$  strictly convex one has that if  $A \in \mathbb{K}^n$  is a maximizer of  $F_{\mathcal{H}^{n-1}}$  then  $h(\cdot) = H(\partial A, \cdot)$  holds on  $\partial A$ . Also, since (cf. [17, Page 197])

$$(n-1)H(\partial A, x) = \Delta b_A(x) \quad \forall \quad x \in \partial A$$

where

$$b_A = d_A - d_{\mathbb{R}^n \setminus A} \quad \& \quad d_E(x) = \text{dist}(x, E) = \min_{y \in E} |x - y| \quad \forall \quad E \in C^n,$$

one concludes that

$$\Delta b_A(x) = (n-1)h(x) \quad \forall \quad x \in \partial A,$$

and so  $b_A$  is of  $C^{k+2,\alpha}$  provided  $h$  is of  $C^{k,\alpha}$ , and consequently,  $\partial A$  is of  $C^{k+2,\alpha}$  due to Delfour-Zolésio's [17, Theorem 5.5].  $\square$

*Remark.* The previous arguments for Theorem 1.1 and its Corollary 1.2, (1.5)-(1.6), the classic variational formula for the volume, and regularities for the Monge-Ampère equations established in [1, 5, 68] can be used to produce a natural Minkowski type proposition – under the hypothesis that  $h \in L^1(\mathbb{R}^n)$  is positive and continuous,  $k$  is a nonnegative integer,  $\alpha \in (0, 1)$ , and

$$F_{\mathcal{L}^n}(A) = \int_A h d\mathcal{L}^n - \mathcal{L}^n(A) \quad \forall \quad A \in C^n,$$

one has:

- $F_{\mathcal{L}^n}(\cdot)$  attains its supremum over  $C^n$  if and only if there exists  $A \in C^n$  such that  $F_{\mathcal{L}^n}(A) \geq 0$ .
- Suppose  $A \in \mathbb{K}^n$  is a maximizer of  $F_{\mathcal{L}^n}(\cdot)$ . Then there is a Borel measure  $\mu_{\mathcal{L}^n, A}$  on  $\mathbb{S}^{n-1}$  such that  $d\mu_{\mathcal{L}^n, A} = \mathbf{g}_*(h d\mathcal{H}^{n-1})$ , namely,

$$\int_{\mathbb{S}^{n-1}} \phi d\mu_{\mathcal{L}^n, A} = \int_{\mathbb{S}^{n-1}} \phi \mathbf{g}_*(h d\mathcal{H}^{n-1}) \quad \forall \phi \in C(\mathbb{S}^{n-1}).$$

In particular, if  $\partial A$  is  $C^2$  strictly convex, then such a maximizer  $A$  satisfies the inverse Gauss curvature equation  $h(\cdot) = (G(\partial A, \cdot))^{-1}$ .

- If  $h$  is of  $C^{k, \alpha}$  and  $A$ , with  $\partial A$  being  $C^2$  strictly convex, is a maximizer of  $F_{\mathcal{L}^n}(\cdot)$ , then  $\partial A$  is of  $C^{k+2, \alpha}$ .

□

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